Student Number: | | | | | | | | |

Teacher Name: ____________________

Penrith Selective High School

2023 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- Reference sheets are provided with this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 74

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 64 marks (pages 6–9)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

	Multiple Choice	Q11	Q12	Q13	Q14	Total
Functions	2, 4 /2	c /5	a /3		e ii /2	/12
Trigonometric Functions	1 /1	b /1		a, b /5	e i, iii /7	/14
Calculus	5, 6, 7, 8 /4	a, d /9	d, e /6		c /4	/23
Combinatorics	9 /1				a, d /3	/4
Proof			c /3			/3
Vectors	3, 10 /2			c, d /10		/12
Binomial Distribution			b /4		b /2	/6
Total	/10	/15	/16	/15	/18	/74

This paper MUST NOT be removed from the examination room

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Which expression is equivalent to $\dfrac{\tan 3\theta - \tan\theta}{1+\tan 3\theta \tan\theta}$?

A. $\tan 4\theta$

B. $\tan 2\theta$

C. $\dfrac{\tan 2\theta}{1+\tan 4\theta}$

D. $\dfrac{\tan\theta}{1+\tan 3\theta}$

2 The polynomial $P(x)=6x^3-2x+bx-4$ has a factor of $2x-1$.
What is the value of b?

A. 8

B. –7.5

C. 8.5

D. –7

3 If $\underset{\sim}{a}=-3\underset{\sim}{i}+\underset{\sim}{j}$ and $\underset{\sim}{b}=-m\underset{\sim}{i}-\underset{\sim}{j}$, where m is a real constant, then the vector $\underset{\sim}{a}-\underset{\sim}{b}$ will be perpendicular to vector $\underset{\sim}{b}$ when:

A. m = –1 or m = –2

B. $m = \dfrac{1}{3}$

C. m = 1 or m = 2

D. $m = \dfrac{11}{3}$

4 Let α, β and γ be the roots of the equation $x^3+2x^2+3x-10=0$.

Find $\dfrac{1}{\alpha\beta}+\dfrac{1}{\beta\gamma}+\dfrac{1}{\alpha\gamma}$.

A. $-\dfrac{1}{5}$

B. $\dfrac{1}{3}$

C. $\dfrac{1}{5}$

D. $-\dfrac{1}{3}$

5 What is the n^{th} derivative of ax^n?

A. $a(n-1)!x$

B. anx^{n-1}

C. $an!x$

D. $an!$

6 If $\int_0^{\frac{\pi}{6}} \dfrac{\cos x}{1-\sin x}dx=\log_e m$, the value of m is:

A. $4+2\sqrt{3}$

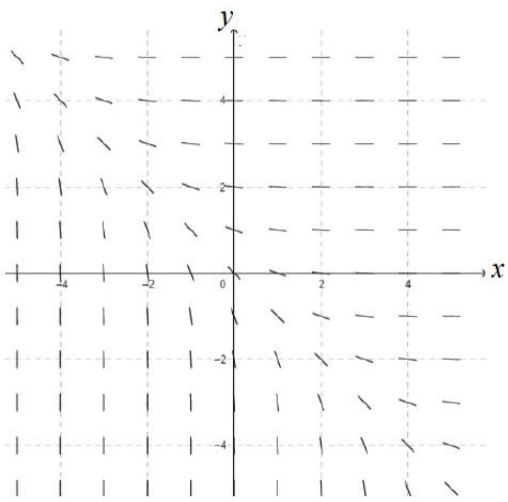
B. $\dfrac{2-\sqrt{3}}{2}$

C. $\dfrac{1}{2}$

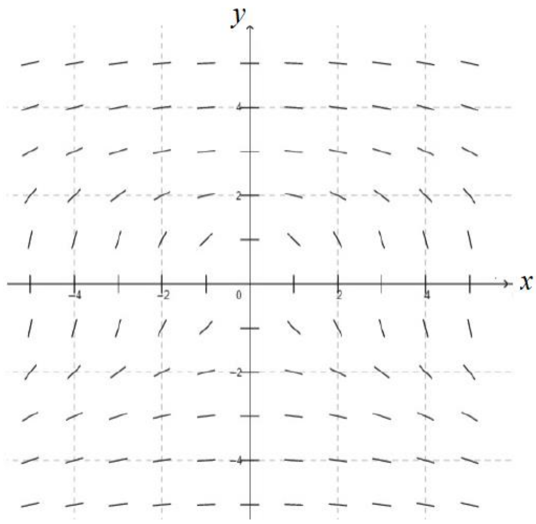
D. 2

7 Which of the following best represents the direction field for the differential equation $\dfrac{dy}{dx} = e^{x-y}$?

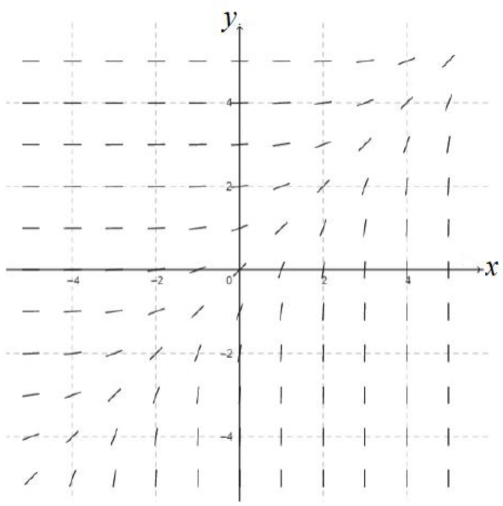
A.



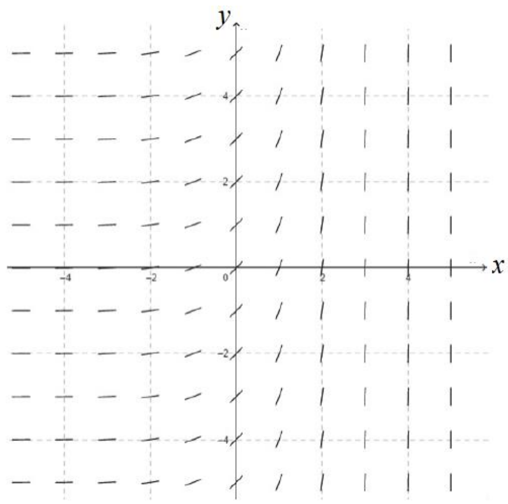
B.



C.



D.



8 Which of the following statements is true for the function $y = 2e^{|x|} - 2$?

A. The function is not continuous at $x = 0$.

B. The function is not differentiable at $x = 0$.

C. The function has an asymptote at $y = -2$.

D. The function has a stationary point at $x = 0$.

9 In how many ways can twelve table tennis players be assigned into three groups each containing four players for the elimination round?

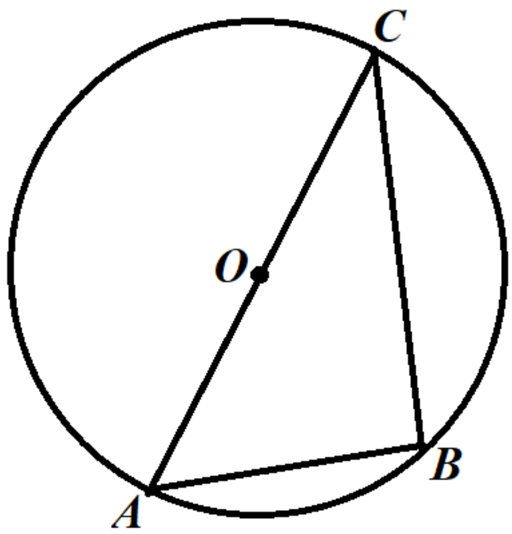
A. 5 775

B. 11 550

C. 15 400

D. 34 650

10 A circle with centre O has a radius $\overrightarrow{OA} = \underset{\sim}{a}$.
B and C are points on the circle and $\overrightarrow{BC} = \underset{\sim}{b}$.



Which one of the following statements must be true?

A. $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{b}$

B. $2\underset{\sim}{a} \bullet \underset{\sim}{b} = -\underset{\sim}{b} \bullet \underset{\sim}{b}$

C. $\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$

D. $\underset{\sim}{a} = -\frac{1}{2}\underset{\sim}{b}$

Section II

64 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet

(a) Find, by integration, the volume of the solid generated when the area under the curve $y = \cos x$ between $x = 0$ and $x = \frac{p}{2}$ is rotated about the x-axis. **3**

(b) State the domain of $y = \frac{3}{4}\cos^{-1} 5x$ using interval notation. **1**

(c) (i) Solve the equation $2x - 9 = \frac{-9}{x}$. **2**

(ii) On the same set of axes, sketch the graph of $y = 2x - 9$ and $y = \frac{-9}{x}$. **1**

(iii) Hence or otherwise, find all the values of x for which $2x - 9 \leq \frac{-9}{x}$. **2**

(d) During a science class, Mr King conducted an experiment where bacteria are grown in a Petri dish. The rate of change of the area of the Petri dish that is covered by the bacteria can be modelled by the differential equation:

$\frac{dA}{dt} = \frac{A}{2}\left(\frac{50 - A}{50}\right)$, where A is the area in cm^2 and t is the time in days.

(i) Show that $\frac{50}{A(50 - A)} = \frac{1}{A} + \frac{1}{50 - A}$. **1**

(ii) Given the initial condition $A(0) = 1$, show that the solution to the differential equation of $\frac{dA}{dt} = \frac{A}{2}\left(\frac{50 - A}{50}\right)$ is $A = \frac{50}{49e^{-0.5t} + 1}$. **4**

(iii) What is the maximum possible area of the bacterium in the Petri dish? **1**

Question 12 (16 marks) Use a separate Writing Booklet

(a) A circle has the equation $x^2 - 4x + y^2 + 8y + 11 = 0$.

(i) Show that it can be written in the form $(x-2)^2 + (y+4)^2 = 9$ **1**

(ii) Express the circle in parametric form. **2**

(b) For a Western ground parrot, there is a chance of 2 in 5 that a fledgling (chick) will survive the first month after hatching. From a brood of a dozen chicks, what is the probability that

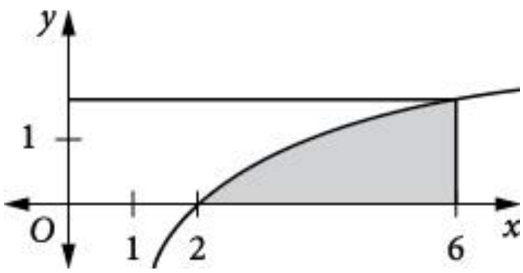
(i) none will survive correct to 3 significant figures? **2**

(ii) more than one will survive correct to 4 decimal places? **2**

(c) Use the principle of mathematical induction to prove that for all integers $n \geq 1$, **3**

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

(d) **2**



The region bounded by the curve $y = \ln|x-1|$, the coordinate axes and the line $y = \ln 5$ is rotated about the y-axis.

Show that the volume of the solid of revolution formed is given by:

$$V = \pi \int_0^{\ln 5} \left(e^{2y} + 2e^y + 1\right) dy.$$

(e) (i) Show that $\dfrac{d}{dx}\left(x \tan^{-1} x\right) = \tan^{-1} x + \dfrac{x}{1+x^2}$. **1**

(ii) Hence, evaluate exactly $\int_0^{\sqrt{3}} \tan^{-1} x \, dx$. **3**

Question 13 (15 marks) Use a separate Writing Booklet

(a) Solve $\sin x \cos x = \dfrac{1}{2}$ for $0 \le x \le 2\pi$. **2**

(b) Solve $3\sin x - \sqrt{3}\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$ **3**

(c) A police officer is testing a batch of bullets imported from China on his shooting range. He holds his rifle at shoulder height of 1.7 m above the ground and shoots horizontally with an initial speed of 340 m s^{-1} at a target 120 m away. (Assume acceleration due to gravity is 10 m s^{-2}.)

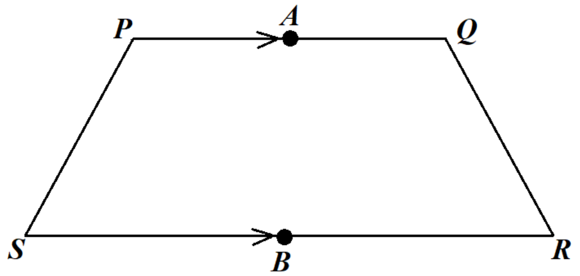
(i) Given that $\ddot{x} = 0$ and $\ddot{y} = -g$, derive the expressions for the velocity and displacement vectors. **2**

(ii) Show that the Cartesian equation is $y = 1.7 - \dfrac{x^2}{23\,120}$. **1**

(iii) Assuming that the ground is horizontal at his range, how far above the ground will the bullet hit the target, to the nearest centimetre? **1**

(iv) Calculate the exact time, in seconds, for the bullet to hit the target and the speed upon impact, in m s^{-1} correct to 2 decimal places. **2**

(d) *PQRS* is a trapezium with *A* and *B* being the midpoints of *PQ* and *RS* respectively.



Let $\overrightarrow{PA} = \underset{\sim}{a}$ and $\overrightarrow{SB} = \underset{\sim}{b}$.

(i) Prove with reasons that $\overrightarrow{QR} = \underset{\sim}{b} - \underset{\sim}{a} + \overrightarrow{AB}$. **2**

(ii) Hence, or otherwise, show that $\overrightarrow{AB} = \dfrac{1}{2}\left(\overrightarrow{PS} + \overrightarrow{QR}\right)$. **2**

Question 14 (18 marks) Use a separate Writing Booklet

(a) In how many ways can an Academic Award, Citizenship Award and Integrity Award be awarded in a class of 24 students? **1**

(b) Four cards are placed face down on a table. The cards are made up of pictures of animals: lion, eagle, wolf and shark. **2**
Nestor bets that he will choose the eagle in a random pick of one of the cards.
If this process is repeated 6 times, express Nestor's success as a binomial random variable and calculate the mean and the variance.

(c) Use the substitution $u = x + 2$ to evaluate $\int_{-2}^{2} x\sqrt{x+2}\ dx$. **4**

(d) By differentiating both sides of the expansion of the expansion of $(1+x)^n$ with respect to *x*, prove that $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + ... + n\binom{n}{n} = n \times 2^{n-1}$. **2**

(e) (i) Prove the trigonometric identity: **3**
$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(ii) Find the quotient of $\left(x^3 - 3x^2 - 3x + 1\right) \div (x+1)$. **2**

(iii) Use the identity from part (e) (i) and let $x = \tan\theta$, to find the roots of the cubic equation $x^3 - 3x^2 - 3x + 1 = 0$ and hence find the exact value of $\tan\frac{\pi}{12}$. **4**

End of paper

2023 Mathematics Extension 1 Trial Examination Solutions

Section 1

① $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

$\therefore \tan(3\theta - \theta) = \tan 2\theta$

$\therefore$ B

② $P(\tfrac{1}{2}) = 6(\tfrac{1}{2})^3 - 2(\tfrac{1}{2}) + b(\tfrac{1}{2}) - 4 = 0$

$\tfrac{3}{4} - 1 + \tfrac{1}{2}b - 4 = 0$

$\therefore b = 8.5$

$\therefore$ C

③ $\underset{\sim}{a} - \underset{\sim}{b} = (-3+m)\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{b} = -m\underset{\sim}{i} - \underset{\sim}{j}$

Since $(\underset{\sim}{a} - \underset{\sim}{b}) \perp \underset{\sim}{b}$

then $(\underset{\sim}{a} - \underset{\sim}{b}) \cdot \underset{\sim}{b} = 0$

$(-3+m)(-m) + 2(-1) = 0$

$3m - m^2 - 2 = 0$

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$\therefore m = 1$ or 2

$\therefore$ C

④ $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\alpha\gamma} = \dfrac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$

$= \dfrac{-2}{10} = -\dfrac{1}{5}$

$\therefore$ A

⑤ let $y = ax^n$

$y' = anx^{n-1}$

$y'' = an(n-1)x^{n-2}$

$y''' = an(n-1)(n-2)x^{n-3}$

$\vdots$

$y^{(n)} = an!\,x^0$

$= an!$

$\therefore$ D

⑥ $-\displaystyle\int_0^{\frac{\pi}{6}} \frac{-\cos x}{1-\sin x}\,dx = \log_e m$

$= -\ln|1-\sin x|\Big]_0^{\frac{\pi}{6}}$

$= -\ln|1-\sin\tfrac{\pi}{6}| + \ln|1-\sin 0|$

$= -\ln\tfrac{1}{2} + 0$

$= \log_e 2$

$\therefore m = 2$

$\therefore$ D

⑦ Given $\dfrac{dy}{dx} = e^{x-y}$

let $x = y = 1$

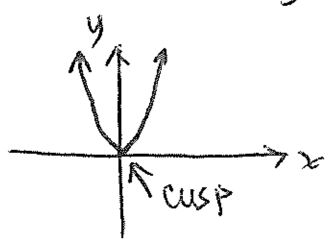
then $\dfrac{dy}{dx} = e^0 = 1$

$\therefore$ when $x = y$, $\dfrac{dy}{dx} = 1$

$\therefore$ C

⑧ Given $y = 2e^{|x|} - 2$

when $x = 0$, $y = 0$

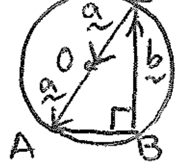


$\therefore$ B (not differentiable at $x = 0$)

⑨ $\dfrac{{}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4}{3!} = 5775$

$\therefore$ A

⑩



$\overrightarrow{BA} = \underset{\sim}{b} + 2\underset{\sim}{a}$

Since $\overrightarrow{BA} \perp \overrightarrow{BC}$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$

$(\underset{\sim}{b} + 2\underset{\sim}{a}) \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{b} \cdot \underset{\sim}{b} + 2\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$2\underset{\sim}{a} \cdot \underset{\sim}{b} = -\underset{\sim}{b} \cdot \underset{\sim}{b}$

$\therefore$ B

Section II

Question 11

(a) Given $y = \cos x$

$V = \pi \int y^2 \, dx$

$V = \pi \int_0^{\pi/2} \cos^2 x \, dx$

$V = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2x) \, dx$

$V = \frac{\pi}{2} \left[x + \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{2}}$

$V = \frac{\pi}{2} \left[\left(\frac{\pi}{2} + \cancel{\tfrac{1}{2} \sin \pi}^{\,0} \right) - \cancel{\left(0 + \tfrac{1}{2} \sin 0 \right)}^{\,0} \right]$

$V = \frac{\pi^2}{4}$ units^3

Overall, all of Q11 was done very well.

(b) Given $y = \frac{3}{4} \cos^{-1} 5x$

Domain: $-1 \leq 5x \leq 1$

$\frac{-1}{5} \leq x \leq \frac{1}{5}$

$\therefore \left[-\frac{1}{5}, \frac{1}{5} \right]$

(c) (i) $2x - 9 = -\frac{9}{x}$

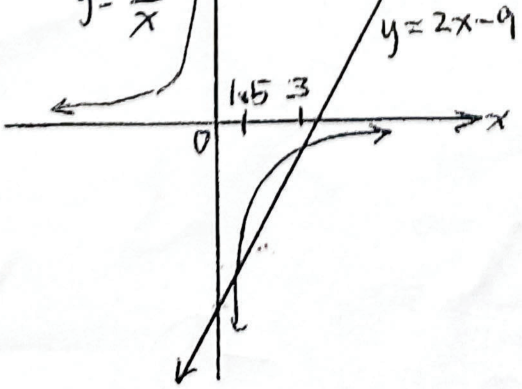
$2x^2 - 9x = -9$

$2x^2 - 9x + 9 = 0$

$(2x - 3)(x - 3) = 0$

$x = \frac{3}{2}$ or 3

(ii)



(iii) $2x - 9 \leq -\frac{9}{x}$

From the graph,

$x < 0$ or $\frac{3}{2} \leq x \leq 3$

Question 11

(d)

(i) $\dfrac{50}{A(50-A)} = \dfrac{1}{A} + \dfrac{1}{50-A}$

$$RHS = \frac{50-A+A}{A(50-A)}$$
$$= \frac{50}{A(50-A)}$$
$$= LHS$$

(ii)
$$\frac{dA}{dt} = \frac{A}{2}\left(\frac{50-A}{50}\right)$$
$$\frac{dt}{dA} = \frac{2}{A}\left(\frac{50}{50-A}\right)$$
$$\frac{dt}{2} = \frac{50\,dA}{A(50-A)}$$
$$\frac{1}{2}\int dt = \int \frac{50}{A(50-A)}\,dA$$
$$\frac{1}{2}t + C = \int\left(\frac{1}{A} + \frac{1}{50-A}\right)dA$$
$$\frac{1}{2}t + C = \ln|A| - \ln|50-A|$$
$$\frac{1}{2}t + C = \ln\left|\frac{A}{50-A}\right|$$
$$\left|\frac{A}{50-A}\right| = e^{\frac{1}{2}t+C}$$
$$\frac{A}{50-A} = e^{\frac{1}{2}t}\cdot e^{C} \quad (\text{since } A>0)$$

let $e^C = K$ where K is a positive constant

$$\frac{A}{50-A} = Ke^{\frac{1}{2}t}$$

when $t=0$, $A=1$

$$\frac{1}{50-1} = Ke^{0}$$
$$\therefore K = \frac{1}{49}$$
$$\frac{A}{50-A} = \frac{e^{\frac{1}{2}t}}{49}$$
$$49A = (50-A)e^{\frac{1}{2}t}$$
$$49A = 50e^{\frac{1}{2}t} - Ae^{\frac{1}{2}t}$$
$$A(49+e^{\frac{1}{2}t}) = 50e^{\frac{1}{2}t}$$
$$A = \frac{(50e^{\frac{1}{2}t}) \div e^{\frac{1}{2}t}}{(49+e^{\frac{1}{2}t}) \div e^{\frac{1}{2}t}}$$
$$A = \frac{50}{49e^{-0.5t}+1}$$

(iii) when $t \to \infty$

$$A = \frac{50}{49(0)+1} = 50$$

$\therefore$ max area $= 50\text{ cm}^2$

Question 12

a) $x^2-4x+y^2+8y+11=0.$

$x^2-4x+4+y^2+8y+16=16+4-11$

(i) (completing the square)

$(x-2)^2+(y+4)^2=9$ ✓

OR.

$(x-2)^2+(y+4)^2=9$

is $x^2-4x+4+y^2+8y+16=9$

$x^2-4x+y^2+8y+20-9=0$ ✓

$x^2-4x+y^2+8y+11=0$

which is the given equation

(ii) using part a(i)

$(x-2)^2+(y+4)^2=3^2$

$\left(\frac{x-2}{3}\right)^2+\left(\frac{y+4}{3}\right)^2=1$ ✓

$\frac{x-2}{3}=\cos\theta, \quad \frac{y+4}{3}=\sin\theta$

as $\cos^2\theta+\sin^2\theta=1$

$x=2+3\cos\theta, \quad y=-4+3\sin\theta$ ✓

(i)
- students successfully completed the square
- Over all well done.

(ii) Many students wrote
- $\sin\theta$ and $\cos\theta$ as subjects and still got full marks.
- few students were rearranging the equation
- overall, not so well done.

b) $P(p) = \frac{2}{5}$, $P(q) = 3/5$, $n = 12$

p - survive, q - does not survive.

(i) $P(X=0) = {}^{12}C_0 (p)^0 q^{12}$ ✓

$= \left(\frac{3}{5}\right)^{12}$

$= 0.0021767$ ✓

$= 0.00218$ (3 SF)

- Many students wrote the correct formula, but answered correct to 3 decimal place, instead of 3 significant figures

(ii) $P(X>1) = 1 - P(X=0) - P(X=1)$ ✓

$= 1 - P(X=0) - {}^{12}C_1 \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^{11}$

$= 1 - \left(\frac{3}{5}\right)^{12} - {}^{12}C_1 \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^{11}$ ✓

$= 0.9804$ (4 dp).

- Many students forgot $P(X=0)$ or $P(X=1)$ and as a result, got an incorrect answer

c) $\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{(n+2)}{2^n}$,

$n \geq 1$

for $n=1$, $LHS = \frac{1}{2}$, $RHS = 2 - \frac{1+2}{2^1}$

$= 2 - \frac{3}{2}$

$= \frac{1}{2}$

$LHS = RHS$

∴ the result is true for $n=1$

- well done for n=1 ✓

- There was only one student who did LHS and RHS together and lost the mark.

Let the result be true for $n=k, k\geq 1$

$\therefore \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+ \cdots + \frac{k}{2^k} = 2-\frac{k+2}{2^k}$ } 1 mark

to prove the result is true for

$n=k+1$

i.e. $\frac{1}{2}+\frac{2}{2^2}+ \cdots + \frac{k}{2^k}+\frac{k+1}{2^{k+1}} = 2-\frac{k+3}{2^{k+1}}$

- very few got mixed up with algebraic operations.

LHS: $\underbrace{\frac{1}{2}+\frac{2}{2^2}+ \cdots + \frac{k}{2^k}}+\frac{k+1}{2^{k+1}}$

$= 2-\frac{k+2}{2^k}+\frac{(k+1)}{2^{k+1}}$ (using assumption)

-1

$= 2-\frac{k+2}{2^k}+\frac{(k+1)}{2^k\times 2}$

- overall it was good

$= 2-\frac{2(k+2)\cancel{+}k+1}{2^{k+1}}$

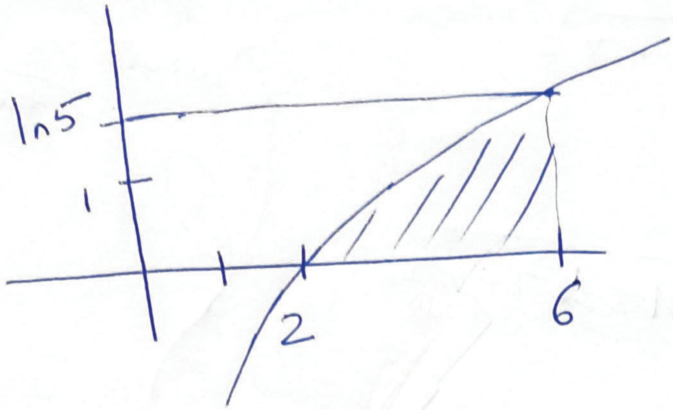
$= 2-\frac{2k+4\cancel{+}k\cancel{+}1}{2^{k+1}}$

$= 2-\frac{k+3}{2^{k+1}}$

$=$ RHS.

$\therefore$ using Principal of Mathematical Induction, the result is true for $\forall n\geq 1$.

d)



common mistake
- forgot to write
π.

$y = \ln|x-1|$

$e^y = |x-1| \Rightarrow$ ~~$x-1 = e^y$,~~

$e^y > 0$

$x = e^y + 1$ ✓

well done.

$V = \pi \int x^2\, dy$ ✓

$= \pi \int_0^{\ln 5} (e^y+1)^2\, dy$

$= \pi \int_0^{\ln 5} (e^{2y} + 2e^y + 1)\, dy$

e) $u = x$ $\quad v = \tan^{-1}x$

$u' = 1$ $\quad v' = \frac{1}{1+x^2}$

few students write $\int \frac{x}{1+x^2}dx = \tan^{-1}x$

(i) $\frac{d}{dx}(uv) = uv' + vu'$

$= \tan^{-1}x + \frac{x}{1+x^2}$

(ii) integrate the above expression

$$\int_0^{\sqrt{3}} \frac{d}{dx}(x\tan^{-1}x)\,dx = \int_0^{\sqrt{3}} \tan^{-1}x\,dx + \int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx.$$

$$\left[x\tan^{-1}x\right]_0^{\sqrt{3}} = \int_0^{\sqrt{3}} \tan^{-1}x\,dx + \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{1+x^2}\,dx \qquad = \int \frac{f'(x)dx}{f(x)}$$

$$\sqrt{3}\times\tan^{-1}\sqrt{3} - 0\times\tan^{-1}0 = \int_0^{\sqrt{3}} \tan^{-1}x\,dx + \frac{1}{2}\left[\ln|1+x^2|\right]_0^{\sqrt{3}}$$

$$\sqrt{3}\times\frac{\pi}{3} - 0 = \int_0^{\sqrt{3}} \tan^{-1}x\,dx + \frac{1}{2}\left[\ln 4 - \ln 1\right]$$

several algebraic errors including $\pm$

$$\frac{\sqrt{3}\pi}{3} - \frac{1}{2}\ln 4 = \int_0^{\sqrt{3}} \tan^{-1}x\,dx$$

OR

$$\frac{\sqrt{3}\pi}{3} - \ln 2 = \int_0^{\sqrt{3}} \tan^{-1}x\,dx.$$

overall, it was okay.

Question 13 (15 marks) Use a separate Writing Booklet

Done well (a) Solve $\sin x \cos x = \frac{1}{2}$ for $0 \le x \le 2\pi$. **2**

$2\sin x \cos x = 1$

$\sin 2x = 1$, $0 \le 2x \le 4\pi$ (1)

$2x = \frac{\pi}{2}, \frac{5\pi}{2}$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$ (1)

* Don't forget to write the new domain, $0 \le 2x \le 4\pi$

Common mistake:

$2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$ Only found one solution.

(b) Solve $3\sin x - \sqrt{3}\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$ **3**

Let $3\sin x - \sqrt{3}\cos x = r\sin(x-\alpha)$

$= r\sin x\cos\alpha - r\cos x\sin\alpha$

Equate coefficients,

$r\cos\alpha = 3$, $r\sin\alpha = \sqrt{3}$

$\cos\alpha = \frac{3}{r}$ [1] $\sin\alpha = \frac{\sqrt{3}}{r}$ [2]

$[1]^2 + [2]^2$,

$\cos^2\alpha + \sin^2\alpha = \frac{9}{r^2} + \frac{3}{r^2}$

$1 = \frac{12}{r^2}$

$r^2 = 12$

$\therefore r = 2\sqrt{3}$ $(r>0)$ (1)

α lies in the 1st quadrant since $\cos\alpha > 0$ & $\sin\alpha > 0$.

From [2], $\sin\alpha = \frac{\sqrt{3}}{2\sqrt{3}}$

$= \frac{1}{2}$

$\therefore \alpha = \frac{\pi}{6}$

$2\sqrt{3}\sin(x - \frac{\pi}{6}) = \sqrt{3}$ (1)

$\sin(x - \frac{\pi}{6}) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore x = \frac{\pi}{3}, \pi$ (1)

* To keep $0 \le \alpha \le \frac{\pi}{2}$, use:

$a\sin x \pm b\cos x = r\sin(x \pm \alpha)$

$a\cos x \pm b\sin x = r\cos(x \mp \alpha)$

* Use the sign of $\cos\alpha = \frac{3}{r}$ and $\sin\alpha = \frac{\sqrt{3}}{r}$ to determine the location of α.

Q.13 b)

<u>Alternative</u> :

Let $t = \tan\frac{x}{2}$

$3\sin x - \sqrt{3}\cos x = \sqrt{3}$

$3\left(\frac{2t}{1+t^2}\right) - \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right) = \sqrt{3}$

$6t - \sqrt{3}(1-t^2) = \sqrt{3}(1+t^2)$

$6t - \sqrt{3} \cancel{+\sqrt{3}t^2} = \sqrt{3} \cancel{+\sqrt{3}t^2}$

$6t = 2\sqrt{3}$

$t = \frac{\sqrt{3}}{3}$

$\tan\frac{x}{2} = \frac{1}{\sqrt{3}}$, $0 \le \frac{x}{2} \le \pi$

$\frac{x}{2} = \frac{\pi}{6}$

$\therefore x = \frac{\pi}{3}$

Check for $x = \pi$, (since t is undefined when $x = \pi$)

$LHS = 3\sin\pi - \sqrt{3}\cos\pi$

$= 0 - \sqrt{3}(-1)$

$= \sqrt{3}$

$= RHS$

$\therefore x = \frac{\pi}{3}, \pi$

Question 13 (15 marks) Use a separate Writing Booklet

(c) A police officer is testing a batch of bullets imported from China on his shooting range. He holds his rifle at shoulder height of 1.7 m above the ground and shoots horizontally with an initial speed of 340 m s^{-1} at a target 120 m away. (Assume acceleration due to gravity is 10 m s^{-2}.)

(i) Given that $\ddot{x} = 0$ and $\ddot{y} = -g$, derive the expressions for the velocity and displacement vectors. **2**

$\underset{\sim}{a} = -10\underset{\sim}{j}$

$\underset{\sim}{v} = \int -10\underset{\sim}{j}\, dt$

$= -10t\underset{\sim}{j} + c$

Sub. $t=0$, $\underset{\sim}{v} = 340\underset{\sim}{i}$

$340\underset{\sim}{i} = c$

$\therefore \underset{\sim}{v} = -10t\underset{\sim}{j} + 340\underset{\sim}{i}$

$\therefore \underset{\sim}{v} = 340\underset{\sim}{i} - 10t\underset{\sim}{j}$ 1

$\underset{\sim}{s} = \int (340\underset{\sim}{i} - 10t\underset{\sim}{j})\, dt$

$= 340t\underset{\sim}{i} - 5t^2\underset{\sim}{j} + D$

Sub. $t=0$, $\underset{\sim}{s} = 1.7\underset{\sim}{j}$

$1.7\underset{\sim}{j} = D$

$\therefore \underset{\sim}{s} = 340t\underset{\sim}{i} - 5t^2\underset{\sim}{j} + 1.7\underset{\sim}{j}$

$\therefore \underset{\sim}{s} = 340t\underset{\sim}{i} + (1.7 - 5t^2)\underset{\sim}{j}$ 1

* Most students derived the equations of the motion in vertical and horizontal components, seperately. Then they did not express the velocity and displacement in vector form. Penalised

* Did not realise $\theta = 0$ when the bullet is fired horizontally. Penalised.

(ii) Show that the Cartesian equation is $y = 1.7 - \dfrac{x^2}{23\,120}$. **1**

From (i), $x = 340t$, $y = 1.7 - 5t^2$ [2]

$t = \dfrac{x}{340}$ [1]

Sub. [1] into [2],

$y = 1.7 - 5\left(\dfrac{x}{340}\right)^2$

$\therefore y = 1.7 - \dfrac{x^2}{23120}$

Question 13 (15 marks) Use a separate Writing Booklet

(c)

(iii) Assuming that the ground is horizontal at his range, how far above the ground will the bullet hit the target, to the nearest centimetre? **1**

From (ii), $y = 1.7 - \frac{x^2}{23120}$

Sub. $x = 120$,

$y = 1.7 - \frac{120^2}{23120}$

$\therefore y = 1.08$ m

(iv) Calculate the exact time, in seconds, for the bullet to hit the target and the speed upon impact, in m s^{-1} correct to 2 decimal places. **2**

$x = 340t$ (from (i))

Sub. $x = 120$,

$120 = 340t$

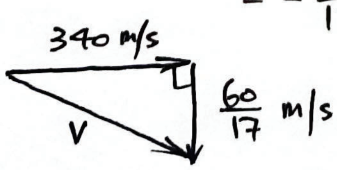
$t = \frac{6}{17}$ 1

When $t = \frac{6}{17}$, $\dot{x} = 340$ and

$\dot{y} = -10t$ (from (i))

$= -10\left(\frac{6}{17}\right)$

$= -\frac{60}{17}$

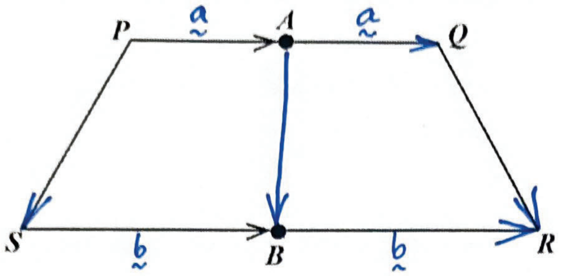


$V = \sqrt{340^2 + \left(-\frac{60}{17}\right)^2}$

$\therefore V = 340.02$ m/s 1

Question 13 (15 marks) Use a separate Writing Booklet

(d) $PQRS$ is a trapezium with A and B being the midpoints of PQ and RS respectively.



Let $\overrightarrow{PA} = \underset{\sim}{a}$ and $\overrightarrow{SB} = \underset{\sim}{b}$.

(i) Prove with reasons that $\overrightarrow{QR} = \underset{\sim}{b} - \underset{\sim}{a} + \overrightarrow{AB}$. **2**

$\overrightarrow{QR} = \overrightarrow{QA} + \overrightarrow{AB} + \overrightarrow{BR}$ ①

$= -\overrightarrow{PA} + \overrightarrow{AB} + \overrightarrow{SB}$ (A is the midpoint of PQ, B is the midpoint of SR) ①

$= -\underset{\sim}{a} + \overrightarrow{AB} + \underset{\sim}{b}$

$\therefore \overrightarrow{QR} = \underset{\sim}{b} - \underset{\sim}{a} + \overrightarrow{AB}$

Alternative 1:

$\overrightarrow{QR} = \overrightarrow{SR} - \overrightarrow{SQ}$

$= 2\overrightarrow{SB} - (\overrightarrow{SB} - \overrightarrow{AB} + \overrightarrow{AQ})$

$= 2\underset{\sim}{b} - (\underset{\sim}{b} - \overrightarrow{AB} + \underset{\sim}{a})$

$= \underset{\sim}{b} + \overrightarrow{AB} - \underset{\sim}{a}$

$\therefore \overrightarrow{QR} = \underset{\sim}{b} - \underset{\sim}{a} + \overrightarrow{AB}$

Alternative 2:

$\overrightarrow{QR} = \overrightarrow{BR} - \overrightarrow{BQ}$

$= \underset{\sim}{b} - (\overrightarrow{AQ} - \overrightarrow{AB})$

$= \underset{\sim}{b} - (\underset{\sim}{a} - \overrightarrow{AB})$

$\therefore \overrightarrow{QR} = \underset{\sim}{b} - \underset{\sim}{a} + \overrightarrow{AB}$

(ii) Hence, or otherwise, show that $\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{PS} + \overrightarrow{QR}\right)$. **2**

$\overrightarrow{PS} = \overrightarrow{PA} + \overrightarrow{AB} + \overrightarrow{BS}$

$= \underset{\sim}{a} + \overrightarrow{AB} - \underset{\sim}{b}$ ①

$RHS = \frac{1}{2}(\overrightarrow{PS} + \overrightarrow{QR})$

$= \frac{1}{2}\left[(\cancel{\underset{\sim}{a}} + \overrightarrow{AB} - \cancel{\underset{\sim}{b}}) + (\cancel{\underset{\sim}{b}} - \cancel{\underset{\sim}{a}} + \overrightarrow{AB})\right]$ ①

$= \frac{1}{2}(2\overrightarrow{AB})$

$= \overrightarrow{AB}$

$= LHS$

Alternative:

$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{SR}$

$\underset{\sim}{b} - \underset{\sim}{a} + \overrightarrow{AB} = -2\underset{\sim}{a} + \overrightarrow{PS} + 2\underset{\sim}{b}$

$\overrightarrow{AB} = -\underset{\sim}{a} + \underset{\sim}{b} + \overrightarrow{PS}$

$= (\overrightarrow{QR} - \overrightarrow{AB}) + \overrightarrow{PS}$ (Since $\overrightarrow{QR} = \underset{\sim}{b} - \underset{\sim}{a} + \overrightarrow{AB}$)

$2\overrightarrow{AB} = \overrightarrow{QR} + \overrightarrow{PS}$

$\therefore \overrightarrow{AB} = \frac{1}{2}(\overrightarrow{PS} + \overrightarrow{QR})$

Question 14

a) Number of ways if same student can have all 3 awards:

$$24^3 = 13824 \quad ①$$

Number of ways if different student:

$$24 \times 23 \times 22 = 12144 \quad ①$$

→ Both answers accepted

→ Done reasonably well, some students did only $^{24}C_3$ without multiplying by 3!

b) Let x = number of eagles chosen

$$X \sim Bin(6, 1/4)$$

$$E(X) = np = 6 \times \frac{1}{4}$$

$$= \frac{3}{2} = 1.5 \quad ①$$

$$Var(X) = np(1-p)$$

$$= 6 \times \frac{1}{4} \times \frac{3}{4}$$

$$= \frac{9}{8} = 1.125 \quad ①$$

→ Done well

→ Students that skipped the question need to go back and revise topic.

c) $$I = \int_{-2}^{2} x\sqrt{x+2}\, dx$$

Let $u = x+2$ $\quad u - 2 = x$

$$\frac{du}{dx} = 1$$

$$= \int_0^4 (u-2)\sqrt{u}\, du \quad ①$$

New bounds:

$u = (2)+2 = 4$ $\quad u = (-2)+2 = 0$ ①

$$= \int_0^4 \left(u^{3/2} - 2u^{1/2}\right) du$$

$$① = \left[\frac{2}{5}u^{5/2} - 2\left(\frac{2}{3}u^{3/2}\right)\right]_0^4$$

$$= \frac{2}{5}(4)^{5/2} - \frac{4}{3}(4)^{3/2} - 0$$

$$= \frac{32}{15} \quad ①$$

→ Done very well

→ few marks were lost due to careless algebra errors.

d) Expanding the expression:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$$

Differeutiating both sides:

$$1 \quad n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^3 + \dots + n\binom{n}{n}x^{n-1}$$

Sub $x=1$,

$$n(1+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2}(1) + 3\binom{n}{3}(1)^3 + \dots + n\binom{n}{n}(1)^{n-1}$$

$$1 \quad n\,2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$$

→ Most students got 1/2

→ Some students couldn't recognise why the "x" disappeared.

e) i)

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

LHS: $\tan(2\theta + \theta)$

$$= \frac{\tan(2\theta) + \tan(\theta)}{1 - \tan(2\theta)\tan(\theta)} \quad 1$$

$$= \frac{\dfrac{2\tan(\theta)}{1-\tan^2(\theta)} + \tan(\theta)}{1 - \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}\tan(\theta)} \quad \frac{\times\ 1-\tan^2(\theta)}{\times\ 1-\tan^2(\theta)}$$

1

$$= \frac{2\tan(\theta) + \tan(\theta)(1-\tan^2(\theta))}{1-\tan^2(\theta) - 2\tan^2(\theta)}$$

$$= \frac{2\tan(\theta) + \tan(\theta) - \tan^3(\theta)}{1 - 3\tan^2(\theta)}$$

$$= \frac{3\tan(\theta) - \tan^3(\theta)}{1 - 3\tan^2(\theta)} \quad 1$$

$=$ RHS

→ Done mostly well, almost all students got 1st marn.

→ Reason for mistalles is incorrect algebra, not being able to read their own hand-writting.

→ Some students skipped too many lines.

e) ii)

$$\begin{array}{r} x^2-4x+1 \\ x+1 \enclose{longdiv}{x^3-3x^2-3x+1} \\ -\underline{\;x^3+x^2+0+0\;} \\ 0-4x^2-3x+1 \\ -\underline{\;-4x^2-4x+0\;} \\ 0+x+1 \\ -\underline{\;x+1\;} \\ 0 \end{array}$$ ①

$\therefore$ Quotient is $\underbrace{x^2-4x+1}_{①}$

→ Done mostly well
→ Biggest mistake: $-3x^2 - x^2 = 2x^2$ ☹

iii) $x^3 - 3x^2 - 3x + 1 = 0$

Let $x = \tan\theta$

$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$

$1 - 3\tan^2\theta = 3\tan\theta - \tan^3\theta$

$\therefore \quad 1 = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ ①

From part i)

$\Rightarrow \tan(3\theta) = 1$

$3\theta = \tan^{-1}(1)$

$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$

$\therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \ldots$

Only the 1st three are needed as they are the ones with distinct solutions. ①

→ Most students got 1/4
→ very few got 4/4
→ Modelled after HSC Q where a mark was allocated to explaining the distinct roots.

$x = \tan\left(\frac{\pi}{12}\right), \tan\left(\frac{5\pi}{12}\right), \tan\left(\frac{3\pi}{4}\right)$

$\tan\left(\frac{3\pi}{4}\right) = -1$ which corresponds to one of the roots of the polynomial.

→ Some used sum and products of roots, very clever! ✓✓

$(x+1)(x^2-4x+1) = 0$

the roots are:

$x = 1, \quad x = \dfrac{4 \pm \sqrt{16-4}}{2}$

$x = 2 \pm \sqrt{3}$ ①

since $\tan\frac{\pi}{12} < \tan\frac{5\pi}{12}$

it follows that:

$\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ①

Mark allocation:

1 - Using part i)

1 - Finding all the distinct solutions and explaining <u>why</u> only 3 are needed.

1 - Finding the other roots $2 \pm \sqrt{3}$

1 - Choosing the correct one for $\tan \pi/12$.